MATH3280A Introductory Probability, 2014-2015
Solutions to HW6

P.365 Ex.2

Solution
(a) The values of the joint probability mass function $p(x, y)$ of X and Y is given by the following table.

$p(x,y)$		y					
		1	2	3	4	5	6
x	2	1/36	0	0	0	0	0
	3	0	2/36	0	0	0	0
	4	0	1/36	2/36	0	0	0
	5	0	0	2/36	2/36	0	0
	6	0	0	1/36	2/36	2/36	0
	7	0	0	0	2/36	2/36	2/36
	8	0	0	0	1/36	2/36	2/36
	9	0	0	0	0	2/36	2/36
	10	0	0	0	0	1/36	2/36
	11	0	0	0	0	0	2/36
	12	0	0	0	0	0	1/36

and $p(x, y) = 0$ elsewhere.

(b) The marginal probability mass function of X is given by

x	2	3	4	5	6	7	8	9	10	11	12
$p_X(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

and $p_X(x) = 0$ elsewhere.

The marginal probability mass function of Y is given by

y	1	2	3	4	5	6
$p_Y(y)$	1/36	3/36	5/36	7/36	9/36	11/36

and $p_Y(y) = 0$ elsewhere.

(c) The expectations of X and Y are

$$E(X) = \sum_{x=2}^{12} x \cdot p_X(x) = 7$$
$$E(Y) = \sum_{y=1}^{6} y \cdot p_Y(y) = \frac{161}{36}$$

□

P.366 Ex.12

Solution
The probability density function of X is

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \begin{cases} \int_0^{1-x} 3(x+y)dy & \text{, if } 0 < x < 1 \\ 0 & \text{, if } x \leq 0 \text{ or } x \geq 1 \end{cases} \\ &= \begin{cases} -\frac{3}{2}x^2 + \frac{2}{3} & \text{, if } 0 < x < 1 \\ 0 & \text{, if } x \leq 0 \text{ or } x \geq 1 \end{cases} \end{aligned}$$

The probability density function of Y is

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx \\ &= \begin{cases} \int_0^{1-y} 3(x+y)dx & \text{, if } 0 < y < 1 \\ 0 & \text{, if } y \leq 0 \text{ or } y \geq 1 \end{cases} \\ &= \begin{cases} -\frac{3}{2}y^2 + \frac{2}{3} & \text{, if } 0 < y < 1 \\ 0 & \text{, if } y \leq 0 \text{ or } y \geq 1 \end{cases} \end{aligned}$$

$$P\left(X + Y < \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} 3(x+y)dydx = \frac{1}{8}$$
$$P\left(X + Y > \frac{1}{2}\right) = \frac{7}{8}$$

□

P.367 Ex.15

Solution
(a) We have

$$1 = \int_0^1 \int_x^1 cx(1-x)dydx = \frac{1}{12}c$$
i.e. $c = 12$.

(b) The joint probability density function of X and Y is

$$f(x,y) = \begin{cases} 12x(1-x) & , \text{if } 0 \le x \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

The marginal probability density function of X is

$$f_X(x) = \begin{cases} \int_x^1 12x(1-x)dy = 12x(1-x)^2 & , \text{if } 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

The marginal probability density function of Y is

$$f_Y(y) = \begin{cases} \int_0^y 12x(1-x)dx = 6y^2 - 4y^3 & , \text{if } 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Since $f(x,y) \neq f_X(x)f_Y(y)$, by Theorem 8.7 X and Y are not independent.
□

P.444 Ex.1

Solution
Let Y be the random variable of the number of tosses until two tails occur successively.
Let

$$X_n = \begin{cases} 1 & \text{, if the n-th toss is tail} \\ 0 & \text{, if the n-th toss is head} \end{cases}$$

$$\begin{aligned} E(Y) &= E(E(Y|X_1)) \\ &= E(Y|X_1 = 0)P(X_1 = 0) + E(Y|X_1 = 1)P(X_1 = 1) \end{aligned}$$

$$\begin{aligned} E(Y|X_1 = 0) &= \sum_{k=2}^{\infty} k \cdot P(Y = k|X_1 = 0) \\ &= \sum_{k=2}^{\infty} k \cdot P(Y = k - 1) \\ &= \sum_{k=1}^{\infty} (k + 1) \cdot P(Y = k) \\ &= \sum_{k=2}^{\infty} (k + 1) \cdot P(Y = k) \\ &= \sum_{k=2}^{\infty} k \cdot P(Y = k) + \sum_{k=2}^{\infty} \cdot P(Y = k) \\ &= E(Y) + 1 \end{aligned}$$

$$E(Y|X_1 = 1) = \sum_{k=2}^{\infty} k \cdot P(Y = k|X_1 = 1)$$

For $k \geq 3$,

$$\begin{aligned} &P(Y = k|X_1 = 1) \\ &= P(Y = k|X_1 = 1, X_2 = 1)P(X_2 = 1) + P(Y = k|X_1 = 1, X_2 = 0)P(X_2 = 0) \\ &= 0 + P(Y = k - 2) \cdot \frac{1}{2} \end{aligned}$$

Then

$$
\begin{aligned}
E(Y|X_1 = 1) &= 2P(Y = 2|X_1 = 1) + \sum_{k=3}^{\infty} k \cdot P(Y = k|X_1 = 1) \\
&= 2 \cdot \frac{1}{2} + \sum_{k=3}^{\infty} k \cdot \frac{1}{2} P(Y = k - 2) \\
&= 1 + \frac{1}{2} \sum_{k=1}^{\infty} (k + 2) \cdot P(Y = k) \\
&= 1 + \frac{1}{2} \sum_{k=2}^{\infty} (k + 2) \cdot P(Y = k) \\
&= 1 + \frac{1}{2} \left(\sum_{k=2}^{\infty} k \cdot P(Y = k) + 2 \sum_{k=2}^{\infty} P(Y = k) \right) \\
&= 1 + \frac{1}{2} \left(E(Y) + 2 \right) \\
&= \frac{1}{2} E(Y) + 2
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= E(E(Y|X_1)) \\
&= E(Y|X_1 = 0)P(X_1 = 0) + E(Y|X_1 = 1)P(X_1 = 1) \\
&= (E(Y) + 1)\frac{1}{2} + (\frac{1}{2}E(Y) + 2)\frac{1}{2} \\
&= \frac{3}{4}E(Y) + \frac{3}{2}
\end{aligned}
$$

Hence $E(Y) = 6$.

□